Imperial College London

Lecture 12

Signal Transmission & Windowing Effects
(Lathi 7.4, 7.6, 7.8)

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Signal Transmission through LTI Systems

- We have seen previously that if x(t) and y(t) are input & output of a LTI system with impulse response h(t), then

$$Y(\omega) = H(\omega)X(\omega)$$

- We can therefore perform LTI system analysis with Fourier transform in a way similar to that of Laplace transform.
- However, FT is more **restrictive** than Laplace transform because the system must be **stable**, and x(t) must itself by Fourier transformable.
- Laplace transform can be used to analyse stable AND unstable system, and apply to signals that grow exponentially.
- If a system is stable, it can shown that the frequency response of the system H(jω) is just the Fourier transform of h(t) (i.e. H(ω)):

$$H(\omega) = H(s)|_{s=j\omega}$$

L7.4 p717

Example

- Find the zero-state response of a stable LTI system with transfer function $H(s) = \frac{1}{s+2}$ and the input is $x(t) = e^{-t}u(t)$.
- The FT of input x(t) is: $X(\omega) = \frac{1}{j\omega + 1}$
- Since the system is stable, therefore H(jω) = H(ω). Hence

$$H(\omega) = H(s)|_{s=j\omega} = \frac{1}{j\omega + 2}$$

- Therefore

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(j\omega+2)(j\omega+1)}$$

- Using partial fractions, we get:

$$Y(\omega) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

L7.4 p717

Time-domain vs Frequency-domain

Impulse response	System response to $e^{j\omega t}$ is $H(w)e^{j\omega t}$
$\delta(t) \Longrightarrow h(t)$	$e^{j\omega t} \Longrightarrow H(\omega)e^{j\omega t}$
x(t) as sum of impulse components	x(t) as sum of everlasting exponential components
$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$
y(t) as sum of responses to impulse components	y(t) as sum of responses to exponential components
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$	$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}\,d\omega$

L7.4 p718

Signal Distortion during transmission

- QUESTION: What is the characteristic of a system that allows signal to pass without distortion?
- Transmission is distortionless if output is identical to input within a multiplicative constant, and relative delay is allowed. That is:

$$y(t) = G_0 x(t - t_d) \quad \Rightarrow \quad Y(\omega) = G_0 X(\omega) e^{-j\omega t_d}$$

- But Y(ω)/X(ω) = H(ω), therefore the frequency characteristic of a distortionless system is:

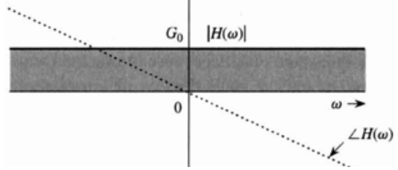
$$H(\omega) = G_0 e^{-j\omega t_d} \quad \begin{cases} |H(\omega)| = G_0 \\ \angle H(\omega) = -\omega t_d \end{cases}$$

For distortionless transmission, amplitude response $|H(\omega)|$ must be a constant AND phase response $\angle H(\omega)$ must be linear function of ω with slope $-t_d$.

L7.4 p720

Group Delay (again)

- $\angle H(\omega) = -\omega t_d$ means that **every** spectral component is delayed by t_d seconds.
- Therefore a distortionless transmission needs a **flat** amplitude response and a **linear** phase response:



- Measure phase linearity with:

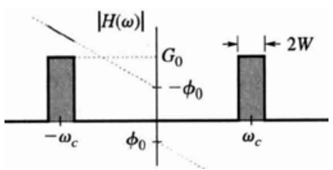
$$t_g(\omega) = -\frac{d}{d\omega}\angle H(\omega)$$

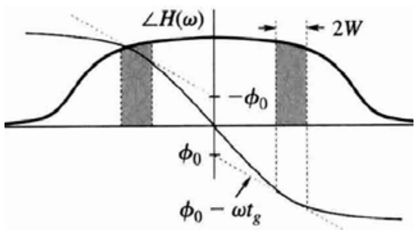
- If $t_g(\omega)$ is constant, signal is delayed by t_g (assuming constant H(ω)).
- $t_g(\omega)$ is known as **Group delay** or **Envelope delay**.
- Human ears are sensitive to amplitude distortion, but not phase distortion.
- Human eyes are sensitive to phase distortion, but not (so much) amplitude distortion

L7.4 p721

Bandpass Systems & Group Delay

- Consider a bandpass system with amplitude and phase characteristics as shown:





- If one applies an input $z(t) = x(t)\cos\omega_c t$, then the output y(t) is:

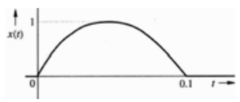
$$y(t) = G_0 x(t - t_g)\cos[\omega_c(t - t_g) + \phi_0)]$$

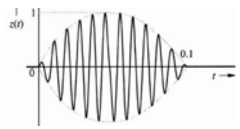
- That is, the output is the delayed version of input z(t) and the output carrier acquires an extra phase ϕ_0.
- The envelope of the signal is therefore distortionless.
- For the proof, see Lathi page 723.

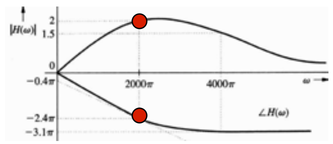
L7.4 p722

Example

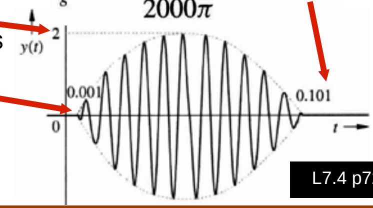
- A signal z(t) shown below is given by $z(t) = x(t)\cos\omega_c t$ where $\omega_c = 2000\pi$. The pulse x(t) is a lowpass pulse of duration 0.1sec and has a bandwidth of about 10Hz. This signal is passed through a filter whose frequency response is shown below. Find and sketch the filter output y(t).







- Z(t) is a narrow band signal with bandwidth of 20Hz centered around 1kHz.
- The gain at 1kHz is 2. The group delay is: $t_g = \frac{2.4\pi - 0.4\pi}{2000\pi} = 10^{-3}$
- The vertical intercept of phase response is $\phi_0 = -0.4\pi$.



L7.4 p724

Parseval's Theorem

- The energy of a signal x(t) can be derived in time or frequency domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

- Proof:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} \, d\omega \right] dt$$

Change order of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right] d\omega$$

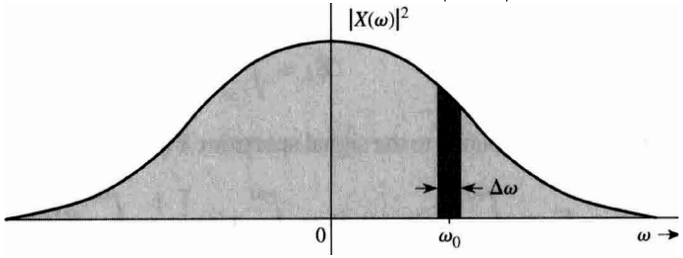
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

L7.6 p730

Energy Spectral Density of a signal

- Total energy is area under the curve of $|X(\omega)|^2$ vs ω (divided by 2π).



- The energy over a small frequency band $\Delta\omega$ ($\Delta\omega \to 0$) is:

$$\Delta E_x = \frac{1}{2\pi} |X(\omega)|^2 \Delta\omega = |X(\omega)|^2 \Delta f \qquad \frac{\Delta\omega}{2\pi} = \Delta f \text{ Hz}$$

Energy spectral density (per unit bandwidth in Hz)

L7.6 p730

Energy Spectral Density of a REAL signal

- If x(t) is a real signal, then $X(\omega)$ and $X(-\omega)$ are conjugate (L11, slide2):

$$|X(\omega)|^2 = X(\omega) X^*(\omega) = X(\omega) X(-\omega)$$

- This implies that $X(\omega)$ is an even function. Therefore

$$E_x = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)|^2 \, d\omega$$

- Consequently, the energy contributed by a real signal by spectral components between ω_1 and ω_2 is:

$$\Delta E_x = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |X(\omega)|^2 \, d\omega$$

L7.6 p730

Example

- Find the energy E of signal $x(t) = e^{-at} u(t)$. Determine the frequency W (rad/s) so that the energy contributed by the spectral component from 0 to W is 95% of the total signal energy E.
- Take FT of x(t): $$X(\omega) = \frac{1}{j\omega + a}$$
- By Parseval's theorem:

$$E_x = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\omega^2 + a^2} \, d\omega = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^{\infty} = \frac{1}{2a}$$

- Energy in band 0 to W is 95% of this, therefore:

$$\frac{0.95}{2a} = \frac{1}{\pi} \int_{0}^{W} \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^{W} = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

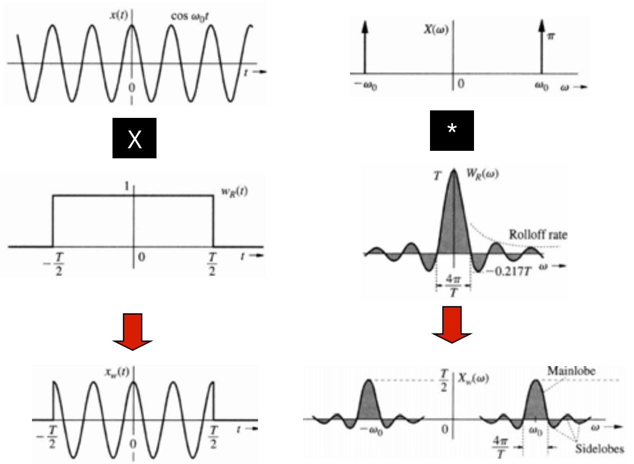
$$\frac{0.95\pi}{2} = \tan^{-1} \frac{W}{a} \implies W = 12.706a \text{ rad/s}$$

- Note: For this signal, 95% of energy is in small frequency band from 0 to 12.7a rad/s or 2.02a Hz!!!

L7.6 p731

Windowing and its effect

- Extracting a segment of a signal in time is the same as multiplying the signal with a rectangular window:



Spectral spreading

Energy spread out from ω0 to width of 2π/T – reduced spectral resolution.

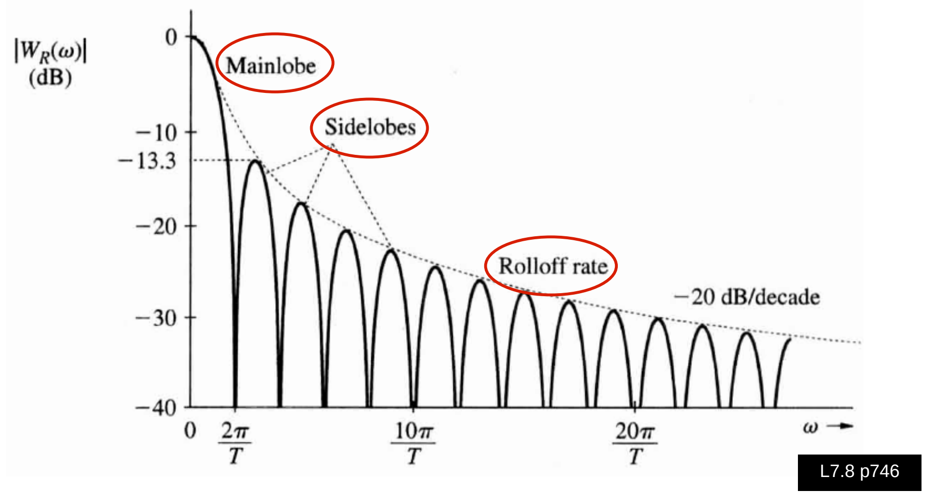
Leakage

Energy leaks out from the mainlobe to the sidelobes.

L7.8 p746

Mainlobe & Sidelobes in dB

- Detail effects of windowing (rectangular window):

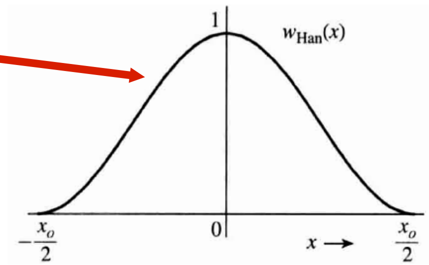


L7.8 p746

Remedies for side effects of truncation

1. Make mainlobe width as narrow as possible -> implies as wide a window as possible.
2. Avoid big discontinuity in the windowing function to reduce leakage (i.e. high frequency sidelobes).
3. 1) and 2) above are incompatible – therefore needs a compromise.

- Commonly used windows outside rectangular window are:
  - Hamming windows
  - Hanning windows
  - Barlett windows
  - Blackman windows
  - Kaiser windows



L7.8 p746

Comparison of different windowing functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	-6	-59.9 $(\alpha = 8.168)$